

## Part IID DIFFERENTIAL GEOMETRY (Mich. 2011): Example Sheet 4

Comments, corrections are welcome at any time.

a.g.kovalev@dpmms.cam.ac.uk.

1. Using geodesic polar coordinates, show that given  $p \in S$  we can express the Gaussian curvature as

$$K(p) = \lim_{r \rightarrow 0} \frac{3(2\pi r - L)}{\pi r^3},$$

where  $L$  is the length of the geodesic circle of radius  $r$  [Hint: Taylor expansion].

2. Let  $\alpha : I \rightarrow S$  be a curve parametrized by arc-length on an oriented surface  $S$ . Show that  $k^2 = k_g^2 + k_n^2$ , where  $k$  is the curvature of  $\alpha$ ,  $k_g$  is its geodesic curvature, and  $k_n$  is the normal curvature.

3. Find the geodesic curvature of a parallel of latitude on the 2-sphere.

4. Prove that on a surface of constant Gaussian curvature, the geodesic circles have constant geodesic curvature.

5. Let  $S$  be a connected surface and  $f, g : S \rightarrow S$  two isometries. Assume that there exists  $p \in S$ , such that  $f(p) = g(p)$  and  $df_p = dg_p$ . Show that  $f(q) = g(q)$  for all  $q \in S$ .

6. (Geodesics are local minimizers of length.) Let  $p$  be a point on a surface  $S$ . Show that there exists an open set  $V$  containing  $p$  such that if  $\gamma : [0, 1] \rightarrow V$  is a geodesic with  $\gamma(0) = p$  and  $\gamma(1) = q$  and  $\alpha : [0, 1] \rightarrow S$  is a regular curve joining  $p$  to  $q$ , then

$$\ell(\gamma) \leq \ell(\alpha)$$

with equality if and only if  $\alpha$  is a reparametrization of  $\gamma$ .

7. Show that in a system of normal coordinates centered at  $p$  (i.e. cartesian coordinates  $(x, y)$  in  $T_p S$  and parametrization  $(x, y) \mapsto \exp_p(xe_1 + ye_2)$ ), all the Christoffel symbols are zero at  $p$ .

8. Let  $S$  be a compact connected orientable surface which is not diffeomorphic to a sphere. Prove that there are points on  $S$  where the Gaussian curvature is positive, negative and zero.

9. Let  $S$  be a compact oriented surface with positive Gaussian curvature and let  $N : S \rightarrow S^2$  be the Gauss map. Let  $\gamma$  be a simple closed geodesic in  $S$ , and let  $A$  and  $B$  be the regions which have  $\gamma$  as a common boundary. Show that  $N(A)$  and  $N(B)$  have the same area.

10. Let  $S$  be an orientable surface with Gaussian curvature  $K \leq 0$ . Show that two geodesics  $\gamma_1$  and  $\gamma_2$  which start from a point  $p \in S$  will not meet again at a point  $q$  in such a way that the traces (i.e. images) of  $\gamma_1$  and  $\gamma_2$  form the boundary of a domain homeomorphic to a disk.

11. Let  $S$  be a surface homeomorphic to a cylinder and with negative Gaussian curvature. Show that  $S$  has at most one simple closed geodesic.

12. Let  $\varphi : U \rightarrow S$  be an orthogonal parametrization around a point  $p$ . Let  $\alpha : [0, \ell] \rightarrow \varphi(U)$  be a simple closed curve parametrized by arc-length enclosing a domain  $R$ . Fix a unit vector  $w_0 \in T_{\alpha(0)} S$  and consider  $W(t)$  the parallel transport of  $w_0$  along  $\alpha$ . Let  $\psi(t)$  be a differentiable determination of the angle from  $\varphi_u$  to  $W(t)$ . Show that

$$\psi(\ell) - \psi(0) = \int_R K dA.$$

Let  $S$  be a connected surface. Use the above to show that if the parallel transport between any two points does not depend on the curve joining the points, then the Gaussian curvature of  $S$  is zero.

The remaining two questions are more ambitious than the previous ones and their content is certainly not examinable. Nevertheless I hope that you will enjoy thinking about some of them.

**13.** (The Poincaré–Hopf theorem.) Let  $S$  be an oriented surface and  $V : S \rightarrow \mathbb{R}^3$  a smooth vector field, that is,  $V(p) \in T_p S$  for all  $p \in S$ . We say that  $p$  is *singular* if  $V(p) = 0$ . A singular point  $p$  is *isolated* if there exists a neighbourhood of  $p$  in which  $V$  has no other zeros. The singular point  $p$  is *non-degenerate* if  $dV_p : T_p S \rightarrow T_p S$  is a linear isomorphism (can you see why  $dV_p$  takes values in  $T_p S$ ?). Show that if a singular point is non-degenerate, then it is isolated.

To each isolated singular point  $p$  we associate an integer called the *index* of the vector field at  $p$  as follows. Let  $\varphi : U \rightarrow S$  be an orthogonal parametrization around  $p$  compatible with the orientation. Let  $\alpha : [0, l] \rightarrow \varphi(U)$  be a regular piecewise smooth simple closed curve, so that  $p$  is the only zero of  $V$  in the domain enclosed by  $\alpha$ . Let  $\psi(t)$  be some differentiable determination of the angle from  $\varphi_u$  to  $V(t) := V \circ \alpha(t)$ . Since  $\alpha$  is closed, there is an integer  $I$  (the index) defined by

$$2\pi I := \psi(l) - \psi(0).$$

(i) Show that  $I$  is independent of the choice of parametrization (Hint: use Problem 12). One can also show that  $I$  is independent of the choice of curve  $\alpha$ , but this is a little harder. Also one can prove that if  $p$  is non-degenerate, then  $I = 1$  if  $dV_p$  preserves orientation and  $I = -1$  if  $dV_p$  reverses orientation.

(ii) Draw some pictures of vector fields in  $\mathbb{R}^2$  with an isolated singularity at the origin. Compute their indices.

(iii) Suppose now that  $S$  is compact and that  $V$  is a smooth vector field with isolated singularities. Consider a triangulation of  $S$  such that

- every triangle is contained in the image of some orthogonal parametrization;
- every triangle contains at most one singular point;
- the boundaries of the triangles contain no singular points and are positively oriented.

Show that

$$\sum_i I_i = \frac{1}{2\pi} \int_S K dA = \chi(S).$$

Thus, you have proved that the sum of the indices of a smooth vector field with isolated singularities on a compact surface is equal to the Euler characteristic (Poincaré–Hopf theorem). Conclude that a surface homeomorphic to  $S^2$  cannot be ‘combed’.

Finally, suppose  $f : S \rightarrow \mathbb{R}$  is a Morse function and consider the vector field given by the gradient of  $f$ , i.e.  $\nabla f(p)$  is uniquely determined by  $\langle \nabla f(p), v \rangle = df_p(v)$  for all  $v \in T_p S$ . Use the Poincaré–Hopf theorem to show that  $\chi(S)$  is the number of local maxima and minima minus the number of saddle points. Use this to determine the Euler characteristic of a surface of genus two.

**14.** (The degree of the Gauss map.) Let  $S$  be a compact oriented surface and let  $N : S \rightarrow S^2$  be the Gauss map. Consider  $y \in S^2$  a regular value. Rather than counting their preimages modulo 2 as we did in the first lectures, we will count them *with sign*. Let  $N^{-1}(y) = \{p_1, \dots, p_n\}$ . Let  $\varepsilon(p_i)$  be  $+1$  if  $dN_{p_i}$  preserves orientation ( $K(p_i) > 0$ ), and  $-1$  if  $dN_{p_i}$  reverses orientation ( $K(p_i) < 0$ ). Now let

$$\deg(N) := \sum_i \varepsilon(p_i).$$

As in the case of the degree mod 2, it can be shown that the sum on the right hand side is independent of the regular value and  $\deg(N)$  turns out to be an invariant of the homotopy class of  $N$ .

Now, choose  $y \in S^2$  such that  $y$  and  $-y$  are regular values of  $N$ . Why can we do so? Let  $V$  be the vector field on  $S$  given by

$$V(p) := \langle y, N(p) \rangle N(p) - y.$$

(i) Show that the index of  $V$  at a zero  $p_i$  is  $+1$  if  $dN_{p_i}$  preserves orientation and  $-1$  if  $dN_{p_i}$  reverses orientation.

(ii) Show that the sum of the indices of  $V$  equals twice the degree of  $N$ .

(iii) Show that  $\deg(N) = \chi(S)/2$ .