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From Fano Threefolds to Compact G_2-Manifolds

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Abstract. G_2-manifolds are 7-dimensional Riemannian manifolds whose metrics have holonomy group G_2; these are necessarily Ricci-flat. We explain a systematic way to construct examples of compact G_2-manifolds by gluing a pair of asymptotically cylindrical manifolds of holonomy $SU(3)$ at their cylindrical ends. To obtain the latter $SU(3)$-manifolds one starts from complex 3-dimensional projective manifolds with $c_1 > 0$ (Fano threefolds) endowed with an appropriate choice of the anticanonical K3 divisor. The resulting G_2-manifolds are topologically distinct from those previously constructed by Joyce.

This article is an informal, introductory account of the 'generalized connected sum' construction of compact Riemannian manifolds with holonomy G_2. Full details and proofs the results can be found in the author's paper [**6**]. A good reference on the Riemannian holonomy groups, including G_2 and the previously known construction of compact G_2-manifolds, is the book by Joyce [**4**].

We briefly review in Section 1 the background results on G_2 holonomy. The method of construction of manifolds of holonomy G_2 is explained in Section 2. Section 3 explains how to obtain examples of this construction using the theory of Fano threefolds and K3 surfaces, and contains a discussion of the results.

1. Synopsis on the holonomy group G_2

The holonomy group $\mathrm{Hol}(g)$ of a Riemannian manifold (M, g) is defined as the group of isometries of the tangent space T_xM generated by parallel transport, using the Levi–Civita connection of g, over closed loops based at x. Up to conjugation, the holonomy group is well-defined as a subgroup of $O(n)$, $n = \dim M$. If M is an oriented simply-connected Riemannian manifold, which is not locally isometric to a Riemannian product or to a Riemannian symmetric space, then there are very few groups which may occur as the holonomy of M, according to Berger's classification theorem. In fact, if in addition one assumes that the dimension of M is *odd* then there are just two possibilities: either $\mathrm{Hol}(g) = SO(n)$ or $\dim M = 7$ and $\mathrm{Hol}(g) = G_2$.

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The group G_2 may be defined as the group of automorphisms of the cross-product algebra on $\mathbb{R}^7$ arising from the identification of $\mathbb{R}^7$ with the purely imaginary octonions. It is a compact Lie group and a (proper) subgroup of $SO(7)$. The cross-product multiplication may be encoded by a 3-form $\varphi_0 \in \Lambda^3(\mathbb{R}^7)^*$,

$$\varphi_0(a,b,c) = \langle a \times b, c\rangle,$$

or, explicitly,

$$\begin{aligned}\varphi_0 =& e_5 \wedge e_6 \wedge e_7 + (e_1 \wedge e_2 + e_3 \wedge e_4) \wedge e_7 \\ &+ (e_1 \wedge e_3 - e_2 \wedge e_4) \wedge e_6 + (e_1 \wedge e_4 + e_2 \wedge e_3) \wedge e_5,\end{aligned} \tag{1.1}$$

where e_i denote an orthonormal basis of $(\mathbb{R}^7)^*$. Conversely, the formula

$$6\langle a,b\rangle\, d\mathrm{vol}_7 = (a\lrcorner\varphi_0) \wedge (b\lrcorner\varphi_0) \wedge \varphi_0. \tag{1.2}$$

expresses the Euclidean inner product in terms of φ_0 and the volume form of $\mathbb{R}^7$. The group G_2 is thus identified as the stabilizer of φ_0 in the natural action of $GL(7,\mathbb{R})$ on $\Lambda^3(\mathbb{R}^7)^*$. The form φ_0 is stable, in the sense of Hitchin [**2**] —the $GL(7,\mathbb{R})$-orbit of φ_0 is open in $\Lambda^3(\mathbb{R}^7)^*$.

A G_2-structure on a 7-manifold, M say, may be given by a 3-form φ such that at each point $p \in M$, $\varphi(p)$ is the image of φ_0 induced by a linear isomorphism $T_pM \to \mathbb{R}^7$. Denote by $\Omega^3_+(M)$ the subset of 3-forms point-wise modelled on φ_0 in the latter sense; elements of $\Omega^3_+(M)$ will sometimes be referred to as the G_2-structure 3-forms. Note that $\Omega^3_+(M)$ is an open subset of $\Omega^3(M)$ in the sup-norm topology, a direct consequence of the stablity property of φ_0.

Every 3-form $\varphi \in \Omega^3_+(M)$ defines an orientation and a Riemannian metric $g = g(\varphi)$ on M, as any G_2-structure is an instance of an $SO(7)$-structure. The formula (1.2) determines $g(\varphi)$ explicitly, up to a conformal factor. The holonomy group of $g(\varphi)$ will be a subgroup of G_2 if and only if the form φ is parallel, $\nabla\varphi = 0$, with respect to the Levi–Civita connection of g. The latter condition is equivalent to the system of partial differential equations on φ [**9**, Lemma 11.5],

$$d\varphi = 0 \qquad \text{and} \qquad d *_\varphi \varphi = 0. \tag{1.3}$$

The second equation in (1.3) is non-linear as the Hodge star $*_\varphi$ is taken in the metric $g(\varphi)$ and depends on φ. The holonomy reduction $\mathrm{Hol}(g(\varphi)) \subseteq G_2$ implies that $g(\varphi)$ is Ricci-flat.

PROPOSITION 1.1 ([**4**, pp.244–245]). *Suppose that a 7-manifold M is compact and let $\varphi \in \Omega^3_+(M)$. Then $\mathrm{Hol}(g(\varphi)) = G_2$ if and only if $\varphi \in \Omega^3_+(M)$ is a solution to* (1.3) and *the fundamental group of M is finite.*

We shall say that a Riemannian 7-manifold (M,g) is a *G_2-manifold* if $\mathrm{Hol}(g) = G_2$ and shall use a similar terminology for other holonomy groups.

The first examples of compact G_2-manifolds were constructed in 1994-5 by Joyce, using a generalized Kummer construction and resolution of singularities. The most elaborate form of this construction can be found in [**4**]. Recently the author obtained different examples of compact G_2-manifolds by a different method [**6**] which we shall now describe.

2. The generalized connected sum construction

Our compact G_2-manifolds are constructed by forming a carefully chosen generalized connected sum of two non-compact Riemannian manifolds with asymptotically cylindrical ends. The construction develops an idea due to Donaldson.

Firstly, we produce a class of complete Ricci-flat Kähler threefolds W of holonomy $SU(3)$ with an infinite cylindrical end asymptotic to the Riemannian product $D \times S^1 \times \mathbb{R}_{>0}$, where D is a K3 surface with a hyper-Kähler metric. This step requires a proof of a non-compact version of the Calabi conjecture, which may be of independent interest.

The Riemannian product $W \times S^1$ carries a solution to (1.3). We consider a pair of such 7-manifolds $W_1 \times S^1$ and $W_2 \times S^1$. For certain pairs of hyper-Kähler K3 surfaces D_i 'at the infinity of W_i' ($i = 1, 2$), there is a way to join the two 7-manifolds $W_i \times S^1$ at their ends to obtain a compact 7-manifold M having finite fundamental group and a 1-parameter family of G_2-structures φ_T compatible with those on $W_i \times S^1$.

A G_2-structure φ_T on M is obtained using cut-off functions, which introduce error terms in the equations (1.3). These error terms are exponentially small in T. We use 'stretching the neck' analysis to prove a gluing theorem, obtaining a solution to (1.3) on M from the solutions on $W_i \times S^1$.

The three parts of the construction are described in more detail below.

2.1. Asymptotically cylindrical Calabi–Yau manifolds. The equations (1.3) define a metric $g(\varphi)$ whose holonomy group is only *contained* in G_2. In particular, the holonomy may be $SU(3)$, a maximal subgroup of G_2.

We begin by introducing the holonomy $SU(n)$ which will be needed in the cases $n = 3$ and $n = 2$. The group $SU(n)$ consists of all the complex linear isomorphisms of $\mathbb{C}^n$ preserving the standard Hermitian inner product and the complex volume. So $SU(n)$ is the stabilizer of the pair of forms on $\mathbb{C}^n$

$$\omega_0 = \tfrac{i}{2}\,(dz_1 \wedge d\bar{z}_1 + \ldots + dz_n \wedge d\bar{z}_n) \qquad \text{and} \qquad \Omega_0 = dz_1 \wedge \cdots \wedge dz_n$$

under the action of $GL(n, \mathbb{C})$. Note that both ω_0 and Ω_0 are stable differential forms (have open orbits under the action of $GL(2n, \mathbb{R})$). A metric g on a real $2n$-manifold Z will have holonomy contained in $SU(n)$ if and only if Z has an $SU(n)$-structure (I, ω, Ω) parallel with respect to g. Here I is an orthogonal complex structure with respect to g, and ω and Ω are differential forms which are point-wise modelled on ω_0 and Ω_0, via a $\mathbb{C}$-linear identification of tangent spaces to Z with $\mathbb{C}^n$. That is to say, a g-parallel $SU(n)$-structure makes Z into a *Kähler complex n-fold* with the Kähler form $\omega \in \Omega^{1,1}(Z)$, and Z has a nowhere vanishing holomorphic form $\Omega \in \Omega^{n,0}(Z)$ such that $\Omega \wedge \Omega^*$ is a constant multiple of ω^n. Such a Ω is sometimes called a holomorphic volume form. In particular, Z has trivial canonical bundle of $(n, 0)$-forms and $c_1(Z) = 0$ and the Kähler metric is Ricci-flat.

Conversely, the following is a direct consequence of Yau's proof of the Calabi conjecture [**12**].

THEOREM 2.1. *Let Z be a Kähler complex n-fold with ω_Z the Kähler form on Z and suppose that $c_1(Z) = 0$. Then there exists on Z a unique Ricci-flat Kähler metric such that its Kähler form is given by $\omega_Z + i\partial\bar{\partial}u$ for some smooth real function u on Z. If Z is simply-connected then the holonomy of this Ricci-flat Kähler metric is contained in $SU(n)$.*

Kähler manifolds with holonomy in $SU(n)$ are often called *Calabi–Yau manifolds*. An important example is a K3 surface; recall that it may be defined as a simply-connected complex surface with $c_1 = 0$. By Yau's theorem, a K3 surface admits a unique Ricci-flat Kähler metric in every Kähler class.

Now let $n = 3$. The group $SU(3)$ is a subgroup of G_2 consisting of all those elements of G_2 which fix a particular one-dimensional subspace in $\mathbb{R}^7$, thus it determines a decomposition $\mathbb{R}^7 \cong \mathbb{C}^3 \oplus \mathbb{R}$. Consider a Kähler threefold W with holonomy in $SU(3)$ and let ω, Ω be respectively the Kähler form and a holomorphic volume form on W. Then on the 7-manifold $W \times S^1$ the 3-form

$$\varphi = \omega \wedge d\theta + \operatorname{Im}\Omega \tag{2.1}$$

is in $\Omega^3_+(W \times S^1)$ and defines a product metric, so $W \times S^1$ has the same holonomy as W. In this metric, one has $*\varphi = \frac{1}{2}\omega \wedge \omega - \operatorname{Re}\Omega \wedge d\theta$ and φ is a solution to (1.3) on $W \times S^1$.

We are now ready to introduce the class of complete $SU(3)$-manifolds that we need. Let $\overline{W}$ be a compact simply-connected Kähler threefold, with $\omega' \in \Omega^{1,1}(\overline{W})$ the Kähler form. Let D be a K3 surface in $\overline{W}$ such that there is a holomorphic section s of the anticanonical bundle $K^{-1}_{\overline{W}}$ vanishing to order 1 on D. It is easy to see that the complement $W = \overline{W} \setminus D$ has trivial canonical bundle.

Assume further that the normal bundle of D in $\overline{W}$ is trivial. Then W can be written as the union of two pieces,

$$W \simeq W_{\text{cpt}} \cup (D \times S^1 \times \mathbb{R}_+) \tag{2.2}$$

a compact manifold W_{cpt} with boundary and a cylindrical end attached along the boundary $D \times S^1$. Note that the relation (2.2) is only a diffeomorphism of the underlying real manifolds. The complex structure on the end of W is *not* isomorphic, but only asymptotic to the 'obvious' product complex structure on $D \times S^1 \times \mathbb{R}_+$.

Let g_D denote the Ricci-flat Kähler metric on D in the Kähler class $[\omega'|_D]$ determined by the embedding in $\overline{W}$. We prove that the following non-compact version of the Calabi conjecture is true.

THEOREM 2.2. *Let $\overline{W}$ and D be as above, so a K3 surface D is an anticanonical divisor and has trivial normal bundle in $\overline{W}$. Suppose also that $\overline{W}$ is simply-connected and the fundamental group of $W = \overline{W} \setminus D$ is finite.*

Then W admits a complete Ricci-flat Kähler metric g_W. The Kähler form and holomorphic volume form of g_W are exponentially asymptotic, along the cylindrical end of W, to those of the product Ricci-flat Kähler structure on $D \times S^1 \times \mathbb{R}_+$ defined using the metric g_D on D. The holonomy of g_W is $SU(3)$.

There is nothing special to threefolds in the proof of Theorem 2.2 and the result extends, with only minor modifications, to Kähler manifolds of arbitrary dimension.

We also remark at this point that previously a number of other non-compact versions of the Calabi conjecture were proved by Tian and Yau, Bando and Kobayashi, and Joyce. These authors construct complete Ricci-flat Kähler metrics asymptotic at infinity to the quotient $\mathbb{C}^n/\Gamma$ of Hermitian $\mathbb{C}^n$ by a finite subgroup Γ of $SU(n)$.

The main novelty of Theorem 2.2 is that it deals with the class of *asymptotically cylindrical* manifolds. We build up on Theorem 5.2 in [**11**] using analysis on exponentially weighted Sobolev spaces to work out the details of asymptotic behaviour and provide control on the boundary data at infinity.

2.2. K3 surfaces and a hyper-Kähler rotation. A remarkable property of a Ricci-flat Kähler metric on a complex surface D is that such a metric is *hyper-Kähler*: the underlying real 4-manifold admits, in addition to the given complex structure I, another complex structure J, such that $IJ = -JI$ and the metric is Kähler with respect to J too. Further, $K = IJ$ is also a complex structure on D, and I, J, K satisfy the quaternionic relations (and define an identification of each tangent space of D with the quaternions). The three respective Kähler forms κ_I,κ_J,κ_K satisfy $\kappa_I^2 = \kappa_J^2 = \kappa_K^2$. There is a complete $SO(3)$ symmetry between I, J, K, in particular, they generate a 2-sphere of complex structures $aI + bJ + cK$ on D, where $a^2 + b^2 + c^2 = 1$, and the metric is Kähler with respect to each of these.

Let D be a Ricci-flat Kähler K3 surface and let κ_I be the Kähler form on D. Then $\kappa_J + i\kappa_K$ defines a holomorphic volume form on D. Considering on D the complex structure J we obtain in general a different Ricci-flat Kähler K3 surface D_J. It has Kähler form κ_J and holomorphic volume form $\kappa_I - i\kappa_K$ and is sometimes called a *hyper-Kähler rotation* of D. Note that there is an S^1-ambiguity in choosing J, as one may take any $bJ + cK$ instead with $b^2 + c^2 = 1$.

Consider two asymptotically cylindrical $SU(3)$-manifolds W_1 and W_2 satisfying the assertions of Theorem 2.2 and, respectively, let D_1,D_2 be the Ricci-flat Kähler K3 surfaces which determine the asymptotic model on the cylindrical ends of W_1, W_2. For $i = 1, 2$, let $t_i \geq 0$ be the real parameter along the cylindrical end of W_i, as defined by (2.2). Cut off at $t_i = T - 1$ the Kähler and holomorphic volume form on each W_i to their asymptotic model on the cylindrical end and consider $W_i(T) \simeq W_{\rm cpt} \cup (D_i \times S^1 \times [0,T])$. Then $W_1(T) \times S^1$ is a manifold with boundary $D_1 \times S^1 \times S^1$ and with a G_2-structure form which on a collar neighbourhood of the boundary is given by

$$\varphi_{(D_1)} = \kappa_I' \wedge d\theta_1 + \kappa_J' \wedge d\theta_2 + \kappa_K' \wedge dt + d\theta_1 \wedge d\theta_2 \wedge dt. \tag{2.3}$$

Here we used (2.1) and the cylindrical asymptotic model $\omega = \kappa_I' + d\theta_2 \wedge dt$, $\Omega = (\kappa_J' + i\kappa_K') \wedge (d\theta_2 + idt)$ of the Kähler and holomorphic volume forms on the end of W_1. In particular, $\varphi_{(D_1)}$ is a solution to (1.3) on the cylinder $(D_1 \times S^1 \times \mathbb{R}) \times S^1$. Similar expressions hold for $W_2 \times S^1$.

Now *assume* that the Ricci-flat Kähler K3 surface D_2 is isomorphic to a hyper-Kähler rotation of D_1. Let $f : D_{1,J} \to D_2$ denote the isomorphism. Then the pull-back action of f on the Kähler forms is given by

$$f^* : \quad \kappa_I'' \mapsto \kappa_J', \quad \kappa_J'' \mapsto \kappa_I', \quad \kappa_K'' \mapsto (-\kappa_K').$$

Define

$$F : (y, \theta_1, \theta_2, t) \in D_1 \times S^1 \times S^1 \times [T-1, T] \to \\ (f(y), \theta_2, \theta_1, 2T - 1 - t) \in D_2 \times S^1 \times S^1 \times [T-1, T] \tag{2.4}$$

and join the two manifolds with boundary to construct a closed oriented 7-manifold

$$M = (W_1(T) \times S^1) \cup_F (W_2(T) \times S^1),$$

using the map F to identify collar neighbourhoods of the boundaries. The compact 7-manifold M is a generalized connected sum with the neck having the cross-section $D \times S^1 \times S^1$. We have $F^*\varphi_{(D_1)} = \varphi_{(D_2)}$, by the construction of F, therefore there is a well-defined 1-parameter family of G_2-structures φ_T on M induced from those on $W_i(T) \times S^1$, defined above using cut-off functions. Here the parameter T is approximately half the length of the neck of M, measured by $g(\varphi_T)$.

The fundamental group of M is *finite*. This is because in the construction of M the circle factor in $W_1(T) \times S^1$ is identified with a circle in W_2 and the circle factor in $W_2(T) \times S^1$ is identified with a circle in W_1, and we assumed that $\pi_1(W_i)$ are finite. Therefore, by Proposition 1.1 any solution $\varphi \in \Omega^3_+(M)$ of the equations (1.3) will define on M a metric $g(\varphi)$ of holonomy G_2.

2.3. The gluing theorem. A G_2-structure form φ_T is constructed by patching the solutions of (1.3) when joining the two pieces of M. This uses cut-off functions which introduce 'error terms' in the equations. In fact we can achieve $d\varphi_T = 0$ for all T and satisfy one of the two equations in (1.3), but the term $d{*_T}\varphi_T$ in general will not vanish. The error terms depend on the difference between the $SU(3)$-structures on the end of W_i and on its cylindrical asymptotic model, and we have an estimate

$$\|d *_T \varphi_T\|_{L^p_k} < C_{p,k} e^{-\lambda T},$$

where $0 < \lambda < 1$. Here $*_T$ denotes the Hodge star of the metric $g(\varphi_T)$.

We prove the following.

THEOREM 2.3. *There exists $T_0 \in \mathbb{R}$ and for every $T \geq T_0$ a unique smooth 2-form η_T on M so that the following holds.*
(1) $\|\eta_T\|_{C^1} < const\cdot e^{-\mu T}$, for some $0 < \mu < 1$, where the C^1-norm is defined using the metric $g(\varphi_T)$. In particular, $\varphi_T + d\eta_T$ is in $\Omega^3_+(M)$.
(2) The closed 3-form $\varphi_T + d\eta_T$ satisfies

$$d *_{\varphi_T + d\eta_T} (\varphi_T + d\eta_T) = 0. \tag{2.5}$$

and so $\varphi_T + d\eta_T$ defines a metric of holonomy G_2 on M.

The equation (2.5) can be rewritten, using the results of [**4**, §10.3] as a non-linear elliptic PDE for η. For small η, this PDE has the form $a(\eta) = a_0 + A\eta + Q(\eta) = 0$, where $a_0 = d *_T \varphi_T$, the linear elliptic operator $A = A_T$ is a compact perturbation of the Hodge Laplacian of the form $dd^* + d^*d + O(e^{-\delta T})$, and $Q(\eta) = O(|d\eta|^2)$.

The central idea in the proof of Theorem 2.3 may be informally stated as follows. For small η, the map $a(\eta)$ is approximated by its linearization and so there is a unique small solution η to the equation $a(\eta) = 0$, for every small a_0 in the range of A. This perturbative approach requires the invertibility of A and a suitable upper bound on the operator norm $\|A_T^{-1}\|$, as $T \to \infty$. This bound determines what is meant by 'small' a_0 in this paragraph.

As we actually need the value of $d\eta$ rather than η, there is no loss in restricting the equation (2.5) for η to the orthogonal complement of harmonic 2-forms on M where the Laplacian is invertible. We use the technique of [**5**, §4.1] based on Fredholm theory for the asymptotically cylindrical manifolds and weighted Sobolev spaces to find an upper bound $\|A_T^{-1}\| < Ge^{\delta T}$. Here the constant G is independent of T and $\delta > 0$ can be taken arbitrary small. So, for large T, the growth of $\|A_T^{-1}\|$ is negligible compared to the decay of $\|d{*_T}\varphi_T\|$ and the 'inverse function theorem' strategy applies to give the required small solution η_T. Standard elliptic methods show that this η_T is in fact smooth.

3. Examples arising from Fano threefolds

For applications of the construction given in Section 2 we need, as a start, to find Kähler threefolds satisfying the hypotheses of Theorem 2.2.

3.1. Introduction to Fano threefolds. The following example is classical in algebraic geometry.

The intersection of three generically chosen quadric hypersurfaces in $\mathbb{C}P^6$ defines a smooth Kähler threefold X_8. It is simply-connected and the characteristic class $c_1(X_8)$ of its anticanonical bundle is the pull-back to X_8 of the positive generator of the cohomology ring $H^*(\mathbb{C}P^6)$. That is to say, the anticanonical bundle $K_{X_8}^{-1}$ is the restriction to X_8 of the tautological line bundle $\mathcal{O}(1)$ over $\mathbb{C}P^6$. It follows that any anticanonical divisor D on X_8 is obtained by taking an intersection $D = X_8 \cap H$ with a hyperplane H in $\mathbb{C}P^6$. A generic such hyperplane section D is a complex surface, isomorphic to a smooth complete intersection of three quadrics in $\mathbb{C}P^5$. This is a well-known example of a K3 surface.

We next look at the normal bundle of X_8 in D. An adjunction-type argument shows that the normal bundle will be trivial if we can find another anticanonical divisor D' on X_8 such that D' does not meet D. But $D' = X_8 \cap H'$ and the second hyperplane section $C = D \cap D' = X_8 \cap H \cap H'$ is *never* empty—it is an algebraic curve (intersection of three quadrics) in $\mathbb{C}P^4$. Fortunately, a suitable threefold can be obtained by blowing up the curve C. The K3 divisor D lifts via the blow-up map $\tilde{X}_8 \to X$ to an isomorphic K3 surface $\tilde{D}$ which is an anticanonical divisor in $\tilde{X}_8$ and has trivial normal bundle. Moreover, a Kähler metric on $\tilde{X}_8$ may be chosen so that $\tilde{D}$ and D are isometric Kähler manifolds.

Finally, as both $\tilde{D}$ and X_8 are simply-connected we find that the only possibility for a nontrivial generator of $\pi_1(X_8 \setminus D)$ would be a circle around $\tilde{D}$. But this circle contracts in an exceptional curve as this curve meets $\tilde{D}$ is exactly one point. Hence $\tilde{X}_8 \setminus \tilde{D}$ is simply connected. The pair $\tilde{X}_8$, $\tilde{D}$ now satisfies all the hypotheses of Theorem 2.2, and so the quasiprojective threefold $W = \tilde{X}_8 \setminus \tilde{D}$ admits an asymptotically cylindrical Ricci-flat Kähler metric of holonomy $SU(3)$.

The threefold X_8 in the above example can be replaced by an arbitrary (smooth) projective-algebraic threefold V with $c_1(V) > 0$, i.e. a *Fano threefold*. Fano threefolds have been extensively studied over the past few decades and a lot is known about them. In particular, they are simply-connected and a generic anticanonical divisor D on a Fano threefold is a K3 surface [**10**]. It can be shown that the threefolds $\tilde{V} \setminus \tilde{D}$ are again simply-connected and we obtain, by application of Theorem 2.2 the following.

PROPOSITION 3.1. *Let V be a Fano 3-fold, $D \in |-K_V|$ a K3 surface, and $\tilde{V}$ the blow-up of V along a self-intersection curve $D \cdot D$, and $\tilde{D}$ the proper transform of D. Then $\tilde{V} \setminus \tilde{D}$ has a complete Ricci-flat Kähler metric with holonomy $SU(3)$. This metric is asymptotic to the Riemannian product $D \times S^1 \times \mathbb{R}_{>0}$, where the Ricci-flat Kähler metric on D is in the Kähler class induced by the embedding in V.*

3.2. Matching the K3 divisors. Let V_1, V_2 be Fano threefolds and D_1, D_2, respectively, anticanonical K3 divisors on these. Recall that by Yau's theorem each of the Kähler K3 surfaces D_i has a uniquely determined Ricci-flat Kähler metric in its Kähler class. If the two Ricci-flat Kähler structures in the Kähler classes of $D_i \subset V_i$ are hyper-Kähler rotations of each other then, in view of Proposition 3.1, we can proceed to the construction of a generalized connected sum M from $\tilde{V}_i \setminus \tilde{D}_i$, as described in Section 2.2. The 7-manifold M admits metrics of holonomy G_2 by Theorem 2.3. In this case, we shall say that a compact G_2-manifold M is *constructed from the pair of Fano threefolds* V_1 *and* V_2. Can we choose D_1 and D_2 so as to satisfy the required hyper-Kähler rotation condition?

We have the freedom to move a K3 surface in the anticanonical linear system of V and to deform V in its algebraic family of Fano threefolds. Recall also from Section 2.2 that there is an S^1-family of choices for the second complex structure J on each D_i. It turns out that, with this freedom, a pair of 'matching divisors' D_1, D_2 can always be found. We now briefly explain, ignoring some important technical points, the ideas in the solution of the matching problem.

All the K3 surfaces are deformations of each other and are diffeomorphic as real 4-manifolds. In particular, their second cohomology lattices are isomorphic to the (unique) even unimodular lattice L of signature $(3, 19)$, known as the K3 lattice. Respectively, $L \otimes \mathbb{C}$ is isomorphic to the second cohomology with complex coefficients and inherits the Hodge decomposition. A Kähler isometry between two K3 surfaces induces a so-called effective Hodge isometry between their second cohomology lattices, preserving the Hodge decomposition and mapping the Kähler class of one K3 to the Kähler class of the other. Surprisingly, the global Torelli theorem for K3 surfaces asserts that the converse is also true: any effective Hodge isometry between second cohomology of two K3 surfaces arises as the pull-back of a unique biholomorphic map between these K3 surfaces [**1**, Ch.VIII]. The latter map will necessarily be an isometry between *Ricci-flat* Kähler K3 surfaces because of the uniqueness of a Ricci-flat Kähler metric in a Kähler class.

We can identify, using a version of the Kodaira–Spencer–Kuranishi deformation theory, the data of Hodge decomposition and Kähler class which occurs in the anticanonical K3 divisors in a given algebraic family of Fano threefolds. The problem of choosing a matching pair of K3 divisors D_i in Fano threefolds V_i then reduces to a problem in the arithmetic of the K3 lattice.

The solution of this problem gives us the following general result.

THEOREM 3.2. *For any pair of algebraic families* $\mathcal{V}_1, \mathcal{V}_2$ *of Fano threefolds there exists (smooth)* $V_1 \in \mathcal{V}_1$, $V_2 \in \mathcal{V}_2$ *such that a compact* G_2*-manifold* M *can be constructed from* V_1, V_2.

This G_2*-manifold satisfies*

$$0 \leq b_2(M) \leq \max\{b_2(V_1), b_2(V_2)\} - 1$$

and

$$b_2(M) + b_3(M) = b_3(V_1) - K_{V_1}^3 + b_3(V_2) - K_{V_2}^3 + 27.$$

EXAMPLE 3.3. A smooth complete intersection X_8 of three quadrics in $\mathbb{C}P^6$ has $b^2 = 1$, $b^3 = 28$, and $-K^3 = 8$. According to Theorem 3.2, an appropriate choice of two such complete intersections $V_8^{(1)}, V_8^{(2)} \subset \mathbb{C}P^6$ and of a hyperplane section D_i in each of the $V_8^{(i)}$ provides data for the construction of a compact G_2-manifold M. We obtain $b_2(M) = 0$ and $b_3(M) = 99$. Also M is simply-connected. This G_2-manifold is not homeomorphic to any of the examples constructed in [**4**].

3.3. Discussion of the results. There is a complete classification of smooth Fano threefolds into 104 algebraic families [**3, 8**]. This provides 5,460 different pairs to form the generalized connected sums, leading to examples of compact G_2-manifolds. As any Fano threefold has $1 \le b^2 \le 10$, we have

$$b_2(M) \le 9,$$

for any G_2-manifold M constructed from a pair of Fanos. Further inspecting the classification list of Fanos, we find that

$$39 \le b_2(M) + b_3(M) \le 239$$

and, in particular, $b_3(M) \ge 30$. (Recall that $b^1 = 0$ for any G_2-manifold, therefore b^2, b^3 determine all the Betti numbers of M.)

It is an interesting question to identify the most general class of Kähler threefolds $\overline{W}$ for which the hypotheses of Theorem 2.2 hold. If the anticanonical linear systems on a pair of such $\overline{W}$ are 'large enough' then the connected sum defined in Section 2 can be formed and will admit G_2-metrics. It seems that the blow-ups $\tilde{V}$ of smooth Fano threefolds discussed in this section can be generalized to include at least manifolds obtained by resolution of singularities in some singular Fano varieties.

A pair of Fano threefolds in general yields several topologically distinct compact G_2-manifolds. Example in [**6**, §8] shows two topologically distinct G_2-manifolds constructed from a pair of $\mathbb{C}P^2 \times \mathbb{C}P^1$'s, realizing both of the values $b_2(M) = 0$ and $b_2(M) = 1$ allowed in this case by Theorem 3.2. Of course, the counting of pairs of Betti numbers (b^2, b^3) only gives a lower estimate of the actual number of topological types realized by our examples of compact G_2-manifolds.

In any event, the majority of smooth Fano threefolds have the Betti number $b^2 \le 4$ and respectively the G_2-manifolds constructed from these have $b^2 \le 3$. On the other hand, a majority of the compact G_2-manifolds constructed in [**4**] have $b^2 > 3$. Thus most of the compact G_2-manifolds constructed from smooth Fano threefolds can be easily identified as new examples, topologically distinct from those previously known.

Another interesting property of the connected sum construction is that it exhibits a new type of boundary point in the moduli space of all G_2-metrics on the given compact 7-manifold M. Any 1-parameter family $\varphi_T + d\eta_T$ of G_2-metrics given by the gluing Theorem 2.3 defines a path in the moduli space. The boundary point attained as $T \to \infty$ corresponds to pulling apart a G_2-manifold at a cross-section K3×(2-torus), obtaining a pair of asymptotically cylindrical pieces. The approach to the boundary of the moduli space in this case involves no development of singularities, nor a curvature growth. This becomes important in [**7**] where we construct the first examples of fibrations of compact G_2-manifolds by certain *minimal* submanifolds called coassociative calibrated submanifolds. The fibrations are an odd-dimensional non-holomorphic analogue of the well-known elliptic fibrations of K3 surfaces.

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