Part III: Differential geometry. Michaelmas 2019 A.G.Kovalev@dpmms.cam.ac.uk

Final schedule.

Local Analysis and Differential Manifolds.

- The concept of a smooth manifold, smooth maps and diffeomorphisms between manifolds. Matrix Lie groups.
- Tangent space to a manifold. The Lie algebra of a matrix Lie group. Tangent and cotangent bundles. Vector fields. Theorem on a trivialization of the tangent bundle.
  The differential of a smooth map. Vector fields as differential operators. Lie bracket (commutator) of vector fields. The Lie algebra of left-invariant vector fields on a Lie group; its isomorphism to a Lie algebra of matrices for a matrix Lie group.
- Embedded submanifolds. Statements of the theorem on the inverse image of a regular value and the Whitney embedding theorem.
- Exterior algebra of differential forms, exterior derivative, pull-back maps. Partition of unity (statement only). Orientation of a manifold. De Rham cohomology, statement of the Poincaré lemma.
  Integration of n-forms over n-dimensional manifolds. Stokes' theorem for manifolds without boundary.

Vector Bundles.

- Definition and examples of vector bundles. Transition functions; G-structures. Principal G-bundles. Constructing bundles from transition functions. The Hopf bundle. Pull-back bundles. Bundle morphisms.
- Differential forms with values in a vector bundle and its endomorphism bundle. Three views on connections: vertical and horizontal subspaces, the local 1-forms $\Gamma^i_{jk}dx^k$, covariant derivatives.
- The curvature form. The (2nd) Bianchi identity.

Riemannian Geometry.

- Connections on manifolds; torsion. Riemannian metrics. The Levi–Civita connection (in local coordinates and a coordinate-free interpretation).
- Horizontal lifts. Basic properties of geodesics. The exponential map. Gauss' Lemma.
- The Riemann curvature. Algebraic symmetries between components of the Riemann curvature. Ricci and scalar curvature.
- The volume form and the Hodge star operator. Laplace–Beltrami operator. Statement of the Hodge decomposition theorem. Harmonic differential forms and de Rham cohomology.

The exam paper will consist of 4 or 5 questions and you will be asked to answer any 3; thus you'll have on average 1 hour per question.

Example sheets form an official part of this course and the exam questions may contain short problem elements similar to what appeared in the examples (*not* those marked with *).

Exam papers of 2001–2019 are at `www.maths.cam.ac.uk/postgrad/mathiii/pastpapers`
Please be aware that there were variations of the course contents; some topics, e.g. geodesics, Laplace–Beltrami operator, were not lectured (thus not examined) in some years. On the other hand, if a past question uses unfamiliar concept(s) then it was probably set on a topic not covered this year (or covered rather differently). Feel free to check with me (e.g. via e-mail) about suitability of particular past questions/papers for revision.